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On topological properties of the Hartman–Mycielski functor

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Abstract. We investigate some topological properties of a normal functor H introduced earlier by Radul which is some functorial compactification of the Hartman–Mycielski construction HM. We prove that the pair (HX, HMY) is homeomorphic to the pair (Q,σ) for each nondegenerated metrizable compactum X and each dense σ-compact subset Y.

Keywords. Hilbert cube; Hartman–Mycielski construction; equiconnected space; normal functor; metrizable compactum; absolute retract.

1. Introduction

The general theory of functors acting on the category $\mathscr{C}omp$ of compact Hausdorff spaces (compacta) and continuous mappings was founded by Shchepin [Sh]. He described some elementary properties of such functors and defined the notion of the normal functor which has become very fruitful. The classes of all normal and weakly normal functors include many classical constructions: the hyperspace exp, the space of probability measures P, the superextension λ, the space of hyperspaces of inclusion G, and many other functors (see [FZ] and [TZ]).

Let X be a space and d an admissible metric on X bounded by 1. By HMX we shall denote the space of all maps from $[0,1)$ to the space X such that $f|[t_i,t_{i+1}) \equiv$ const, for some $0=t_0\le\cdots\le t_n=1$, with respect to the following metric:

$$d_{\mathrm{HM}}(f,g)=\int_0^1 d(f(t),g(t))\mathrm{d}t, \quad f,g\in \mathrm{HM}X.$$

The construction of HMX is known as the Hartman–Mycielski construction [HM]. Recall, that the Hilbert cube is denoted by Q, and the following subspace of Q

$$\{(a_n)_{n=1}^{\infty}\in Q \mid a_k=0 \text{ for all but finitely many } k\}$$

is denoted by σ. Telejko has shown in [Te] that for any nondegenerated separable metrizable σ-compact strongly countable-dimensional space X the space HMX is homeomorphic to σ.

For every $Z \in \mathscr{C}\wr\Updownarrow_{\surd}$ consider

$$\mathrm{HM}_n Z = \big\{ f \in \mathrm{HM}Z \mid \text{there exist } 0 = t_1 < \cdots < t_{n+1} = 1 \\ \text{with } f|[t_i, t_{i+1}) \equiv z_i \in Z, i = 1, \ldots, n \big\}.$$

Let $\mathscr{U}$ be the unique uniformity of Z. For every $U \in \mathscr{U}$ and $\varepsilon > 0$, let

$$\langle \alpha, U, \varepsilon \rangle = \{ \beta \in \mathrm{HM}_n Z \mid m\{ t \in [0,1) \mid (\alpha(t), \beta(t')) \notin U \} < \varepsilon \}.$$

The sets $\langle \alpha, U, \varepsilon \rangle$ form a base of a compact Hausdorff topology in $\mathrm{HM}_n Z$. Given a map $f : X \to Y$ in $\mathscr{C}\wr\Updownarrow_{\surd}$, define a map $\mathrm{HM}_n X \to \mathrm{HM}_n Y$ by the formula $\mathrm{HM}_n F(\alpha) = f \circ \alpha$. Then HM_n is a normal functor in $\mathscr{C}\wr\Updownarrow_{\surd}$ (see [TZ], §2.5.2).

For $X \in \mathscr{C}\wr\Updownarrow_{\surd}$ we consider the space HMX with the topology described above. In general, HMX is not compact. Zarichnyi has asked if there exists a normal functor in $\mathscr{C}\wr\Updownarrow_{\surd}$ which contains all functors HM_n as subfunctors [TZ]. Such a functor H was constructed in [Ra].

We investigate some topological properties of the space HX which is some natural compactification of HMX. The main results of this paper are as follows:

Theorem 1.1. *HX is homeomorphic to the Hilbert cube for each nondegenerated metrizable compactum X.*

Theorem 1.2. *The pair $(HX, \mathrm{HM}Y)$ is homeomorphic to the pair (Q, σ) for each nondegenerated metrizable compactum X and each dense σ-compact strongly countable-dimensional subset Y.*

2. Construction of *H*

Let $X \in \mathscr{C}\wr\Updownarrow_{\surd}$. By $C(X)$ we denote the Banach space of all continuous functions $\varphi : X \to \mathbb{R}$ with the usual sup-norm: $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}$. We denote the segment $[0,1]$ by I.

For $X \in \mathscr{C}\wr\Updownarrow_{\surd}$ let us define the uniformity of HMX. For each $\varphi \in C(X)$ and $a, b \in [0,1]$ with $a < b$ we define the function $\varphi_{(a,b)} : \mathrm{HM}X \to \mathbb{R}$ by

$$\varphi_{(a,b)} = \frac{1}{(b-a)} \int_a^b \varphi \circ \alpha(t) \mathrm{d}t.$$

Define

$$S_{\mathrm{HM}}(X) = \{ \varphi_{(a,b)} \mid \varphi \in C(X) \quad \text{and} \quad (a,b) \subset [0,1) \}.$$

For $\varphi_1, \ldots, \varphi_n \in S_{\mathrm{HM}}(X)$ define a pseudometric $\rho_{\varphi_1,\ldots,\varphi_n}$ on HMX by the formula

$$\rho_{\varphi_1,\ldots,\varphi_n}(f,g) = \max\{ |\varphi_i(f) - \varphi_i(g)| \mid i \in \{1, \ldots, n\} \},$$

where $f, g \in \mathrm{HM}X$. The family of pseudometrics

$$\mathscr{P} = \{ \rho_{\varphi_\infty,\ldots,\varphi_\setminus} \mid \setminus \in \mathbb{N}, \quad \text{where } \varphi_\infty, \ldots, \varphi_\setminus \in \mathscr{S}_{\mathrm{HM}}(\mathscr{X}) \},$$

defines a totally bounded uniformity $\mathscr{U}_{\mathrm{HM}\mathscr{X}}$ of HMX [Ra].

For each compactum X we consider the uniform space $(HX, \mathscr{U}_{\mathscr{H}\mathscr{X}})$ which is the completion of $(\mathrm{HM}X, \mathscr{U}_{\mathrm{HM}\mathscr{X}})$ and the topological space HX with the topology induced by the uniformity $\mathscr{U}_{\mathscr{H}\mathscr{X}}$. Since $\mathscr{U}_{\mathrm{HM}\mathscr{X}}$ is totally bounded, the space HX is compact.

Let $f\colon X \to Y$ be a continuous map. Define the map $\mathrm{HM}f : \mathrm{HM}X \to \mathrm{HM}Y$ by the formula $\mathrm{HM}f(\alpha) = f \circ \alpha$, for all $\alpha \in \mathrm{HM}X$. It was shown in [Ra] that the map $\mathrm{HM}f : (\mathrm{HM}X, \mathscr{U}_{\mathrm{HM}\mathscr{X}}) \to (\mathrm{HM}\mathscr{Y}, \mathscr{U}_{\mathrm{HM}\mathscr{Y}})$ is uniformly continuous. Hence there exists the continuous map $Hf : HX \to HY$ such that $Hf|\mathrm{HM}X = \mathrm{HM}f$. It is easy to see that $H : \mathscr{C}\wr\Updownarrow_{\surd} \to \mathscr{C}\wr\Updownarrow_{\surd}$ is a covariant functor and HM_n is a subfunctor of H for each $n \in \mathbb{N}$.

3. Preliminaries

All spaces are assumed to be metrizable and separable. We begin this section with the investigation of certain structures of equiconnectivity on HX for some compactum X. The first one will be the usual convexity.

Let us remark that the family of functions $S_{\mathrm{HM}}(X)$ embed HMX in the product of closed intervals $\prod_{\varphi_{(a,b)} \in S_{\mathrm{HM}}(X)} I_{\varphi_{(a,b)}}$, where $I_{\varphi_{(a,b)}} = [\min_{x\in X} |\varphi(x)|, \max_{x\in X} |\varphi(x)|]$. Thus, the space HX is the closure of the image of HMX. We denote by $p_{\varphi_{(a,b)}} : HX \to I_{\varphi_{(a,b)}}$ the restriction of the natural projection.

Lemma 3.1. *HX is a convex subset of* $\prod_{\varphi_{(a,b)} \in S_{\mathrm{HM}}(X)} I_{\varphi_{(a,b)}}$.

Proof. It is enough to prove that for each α, $\beta \in \mathrm{HM}X \subset \prod_{\varphi_{(a,b)} \in S_{\mathrm{HM}}(X)} I_{\varphi_{(a,b)}}$ we have that $\frac{1}{2}\alpha + \frac{1}{2}\beta \in HX$.

Let $0 = r_0 < r_1 < \cdots < r_k = 1$ and $0 = p_0 < p_1 < \cdots < p_m = 1$ be the decompositions of the unit interval which correspond to α and β. Consider any $\varphi^1, \ldots, \varphi^n \in C(X)$ and $(a_i, b_i) \subset (0,1)$ for $i \in \{1, \ldots, n\}$. Choose a decomposition of the unit interval $0 = t_0 < t_1 < \cdots < t_s = 1$ such that $\{r_0, \ldots, r_k\} \cup \{p_0, \ldots, p_m\} \cup \{a_i | i \in \{1, \ldots, n\}\} \cup \{b_i | i \in \{1, \ldots, n\}\} \subset \{t_0, \ldots, t_s\}$.

Define $\gamma \in \mathrm{HM}X$ as follows. Let $t \in [t_l, t_{l+1})$ for some $l \in \{0, \ldots, s-1\}$. Define $\gamma(t) = \alpha(t)$ if $t < \frac{1}{2}(t_l + t_{l+1})$ and $\beta(t)$ otherwise. It is easy to see that $p_{\varphi^i_{(a_i,b_i)}}(\gamma) = p_{\varphi^i_{(a_i,b_i)}}(\frac{1}{2}\alpha + \frac{1}{2}\beta)$ for each $i \in \{1, \ldots, n\}$. Hence $\frac{1}{2}\alpha + \frac{1}{2}\beta \in \mathrm{Cl}(\mathrm{HM}X) = HX$. The lemma is proved.

COROLLARY 3.2.

HX is absolute retract for each metrizable compactum X.

Define the map $e_1 : \mathrm{HM}X \times \mathrm{HM}X \times I \to \mathrm{HM}X$ by the condition that $e_1(\alpha_1, \alpha_2, t)(l)$ is equal to $\alpha_1(l)$ if $l < t$ and $\alpha_2(l)$ in the opposite case for $\alpha_1, \alpha_2 \in \mathrm{HM}X$, $t \in I$ and $l \in [0,1)$. We consider HMX with the uniformity $\mathscr{U}_{\mathrm{HM}\mathscr{X}}$ and I with the natural metric.

Lemma 3.3. *The map $e_1 : HMX \times HMX \times I \to HMX$ is uniformly continuous.*

Proof. Let us consider any $U \in \mathscr{U}_{\mathrm{HM}\mathscr{X}}$. We can suppose that

$$U = \{(\alpha, \beta) \in \mathrm{HM}X \times \mathrm{HM}X \mid |\varphi_{(0,1)}(\alpha) - \varphi_{(0,1)}(\beta)| < \delta\},$$

for some $\delta > 0$ and $\varphi \in C(X)$. The proof of the general case is the same.

Put $c = \max_{x\in X} |\varphi(x)|$. Choose $n \in \mathbb{N}$ such that $1/n < \delta/2c$ and put $a_i = i/n$ for $i \in \{0,\dots,n\}$. Consider an element V of uniformity $\mathscr{U}_{\mathrm{HM}\mathscr{X}}$ defined as follows:

$$V = \Big\{(\alpha,\beta) \in \mathrm{HM}X \times \mathrm{HM}X \mid |\varphi_{(a_i,a_{i+1})}(\alpha) - \varphi_{(a_i,a_{i+1})}(\beta)| < \frac{\delta}{2n^2},$$
$$\text{for each } \; i \in \{0,\dots,n-1\}\Big\}.$$

Put $E = \{(l,s) \in I \times I \mid |l-s| < \frac{1}{2n}\}$.

Let us consider any $((\alpha_1,\beta_1),(\alpha_2,\beta_2),(l,s)) \in V \times V \times E$. Then we have

$$\begin{aligned}
&\left|\int_{a_i}^{a_{i+1}} (\varphi\circ\alpha_1(t) - \varphi\circ\alpha_2(t))\mathrm{d}t\right| \\
&\quad = n\frac{1}{a_{i+1}-a_i}\left|\int_{a_i}^{a_{i+1}} (\varphi\circ\alpha_1(t) - \varphi\circ\alpha_2(t))\mathrm{d}t\right| \\
&\quad = n|\varphi_{(a_i,a_{i+1})}(\alpha_1) - \varphi_{(a_i,a_{i+1})}(\alpha_2)| < n\frac{\delta}{2n^2} = \frac{\delta}{2n},
\end{aligned}$$

for each $i \in \{0,\dots,n-1\}$. We have the same for β_1 and β_2. Since $|t_1 - t_2| < \frac{1}{2n}$, there exists $i_0 \in \{0,\dots,n-1\}$ such that $t_1,t_2 \in [a_{i_0}, a_{i_0+1}]$. Then we have

$$\begin{aligned}
&|\varphi_{(0,1)}(e_1(\alpha_1,\beta_1,t_1)) - \varphi_{(0,1)}(e_1(\alpha_2,\beta_2,t_2))| \\
&\quad = \left|\int_0^1 (\varphi\circ e_1(\alpha_1,\beta_1,t_1)(t) - \varphi\circ e_1(\alpha_2,\beta_2,t_2)(t))\mathrm{d}t\right| \\
&\quad \le \sum_{i=0}^{i_0-1}\left|\int_{a_i}^{a_{i+1}} (\varphi\circ\alpha_1(t) - \varphi\circ\alpha_2(t))\mathrm{d}t\right| + c(a_{i_0+1} - a_{i_0}) \\
&\qquad + \sum_{i=i_0}^{n-1}\left|\int_{a_i}^{a_{i+1}} (\varphi\circ\beta_1(t) - \varphi\circ\beta_2(t))\mathrm{d}t\right| < \frac{n-1}{2n}\delta + \frac{\delta}{2} < \delta.
\end{aligned}$$

Therefore $(e_1(\alpha_1,\beta_1,t_1), e_1(\alpha_2,\beta_2,t_2)) \in V$. Thus the lemma is proved.

Hence there exists the extension of e_1 to the continuous map $e : HX \times HX \times I \to HX$. It is easy to check that $e(\alpha,\alpha,t) = \alpha$, for each $\alpha \in HX$.

4. Proofs

Proof of Theorem 1.1. Since HX is infinite-dimensional convex compactum, Theorem 1.1 follows by Klee Theorem [Kl].

We will use the equiconnectivity defined by the function e to prove Theorem 1.2. Let (Z,e) be an equiconnected space where $e : Z \times Z \times I \to Z$ is the map which defines the structure of equiconnectedness. A subset $A \subset Z$ is called *e-convex* if $e(a,b,t) \in A$, for each a, $b \in A$ and $t \in I$.

Consider any $n \in \mathbb{N}$. Let D_{n-1} be a standard $n-1$-simplex in $\mathbb{R}^n$. Define a map $e_n : Z^n \times D_{n-1} \to Z$ as follows. Consider any $(a_1,\dots,a_n,\lambda_1,\dots,\lambda_n) \in Z^n \times D_{n-1}$. Define the finite

sequence $\{x_1,\dots,x_n\}$ by induction. Put $x_1 = a_1$. Suppose that we have already defined x_j for each $j \le i-1$ where $1 < i \le n$. Define

$$x_i = e\left(x_{i-1}, a_i, \frac{\sum_{l=1}^{i-1}\lambda_l}{\sum_{l=1}^{i}\lambda_l}\right).$$

Put $e_n(a_1,\dots,a_n,\lambda_1,\dots,\lambda_n) = x_n$. One can check that the function e_n is continuous.

Let us recall that $A \subset Z$ is homotopically dense in Z if there exists a homotopy $H : Z \times I \to Z$ such that $H(Z \times (0,1]) \subset A$.

Lemma 4.1. *Let (Z,e) be an equiconnected absolute retract and let $A \subset Z$ be a dense e-convex subset. Then A is homotopically dense in Z.*

Proof. It is enough to prove that for every $n \in \mathbb{N}$, every point $z \in Z$ and every neighborhood $U \subset Z$ of z there exists a neighborhood V of z such that every map $f : \partial I^n \to V \cap A$ admits an extension $F : I^n \to U \cap A$ [To].

Since $e_{n+1}(z,\dots,z,\lambda_1,\dots,\lambda_{n+1}) = z$ for each $(\lambda_1,\dots,\lambda_{n+1}) \in D_n$, there exists a neighborhood V of z such that $e_{n+1}(z_1,\dots,z_{n+1},\lambda_1,\dots,\lambda_{n+1}) \in U$ for each $z_1,\dots,z_{n+1} \in V$ and $(\lambda_1,\dots,\lambda_{n+1}) \in D_n$.

Consider any map $f : \partial I^n \to V \cap A$. Choose any metric d on I^n. Let us consider a Dugunji system for $I^n \setminus \partial I^n$, that is, an indexed family $\{U_s, a_s\}_{s\in S}$ such that

(1) $U_s \subset I^n \setminus \partial I^n$, $a_s \in \partial I^n$ $(s \in S)$,
(2) $\{U_s\}_{s\in S}$ is a locally finite cover of $I^n \setminus \partial I^n$,
(3) If $x \in U_s$, then $d(x,a_s) \le 2d(x,\partial I^n)$ for $s \in S$ (see of Ch. II, §3 of [BP]) for more details.

Since $\dim I^n = n$, we can suppose that for each $x \in I^n \setminus \partial I^n$ there exists a neighborhood of X which meets at most $n+1$ elements of $\{U_s\}_{s\in S}$. Moreover we can suppose that the index set is countable. Choose some partition of unity $\{b_s\}_{s\in S}$ inscribed into $\{U_s\}_{s\in S}$. Fix some order $S = \{s_1, s_2, \dots\}$. Define a function $F : I^n \to U$ as follows. Put $F(x) = f(x)$ for each $x \in \partial I^n$. Now consider any $x \in I^n \setminus \partial I^n$. There exists a finite sequence $(s_{i_1},\dots,s_{i_{n+1}})$ such that $x \notin U_s$ for each $s \in S \setminus \{s_{i_1},\dots,s_{i_{n+1}}\}$. Put $F(x) = e_{n+1}(f(a_{s_{i_1}}),\dots,f(a_{s_{i_{n+1}}}), b_{s_{i_1}}(x),\dots,b_{s_{i_{n+1}}}(x))$. One can check that the function F is a continuous extension of the function f. Since A is e-convex, it follows that $f(I^n) \subset A$. The lemma is thus proved.

Proof of Theorem 1.2. It is easy to check that HMX is an e-convex subset of HX. Since HX is an equiconnected absolute retract, and HMY is a dense e-convex subset, homeomorphic to σ [Te], Theorem 1.2 follows by [BRZ]; Proposition 3.1.7 and Lemma 4.1.

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